8.2 – Compositions and Inverse Transformations

Definition: If $T : V \rightarrow W$ is a linear transformation from a vector space V to a vector space W, then T is said to be **one-to-one** if T maps distinct vectors in V into distinct vectors in W.
OR $T : V \rightarrow W$ is one-to-one if and only if for each vector $\mathbf{w}$ in the range of T, there is exactly one vector $\mathbf{v}$ in V such that $T(\mathbf{v}) = \mathbf{w}$.
OR $T : V \rightarrow W$ is one-to-one if and only if $T(\mathbf{u}) = T(\mathbf{v})$ implies $\mathbf{u} = \mathbf{v}$.

Definition: If $T : V \rightarrow W$ is a linear transformation from a vector space V to a vector space W, then T is said to be **onto** (or **onto W**) if every vector in W is the image of at least one vector in V.

Examples:

One-to-one and onto

- Rotation operators on R^2 $(0 \leq \theta < 2\pi)$
- $T: P_3 \to R^4$ where $T(a+bx+cx^2+dx^3) = (a,b,c,d)$
- $T: M_{22} \to R^4$ where $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a,b,c,d)$

One-to-one but not onto

- $T: P_n \to P_{n+1}$, where $T(\vec{p}) = T(p(x)) = x\,p(x)$

Not one-to-one

- $D: C^1(-\infty,\infty) \to F(-\infty,\infty)$, where
  $D(\vec{f}) = f'(x)$ Ex: $D(x^2) = D(x^2+1) = 2x$

Theorem 8.2.1 If $T : V \to W$ is a linear transformation, then T is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$.

pf: ($\Rightarrow$) By Thm 8.1.1, $T(\vec{0}) = \vec{0}$ because T is linear. If T is one-to-one, then if $T(\vec{v}) = \vec{0}$, then $\vec{v} = \vec{0}$.

($\Leftarrow$) Let $\vec{u}, \vec{v} \in V$ such that $\vec{u} \neq \vec{v}$.

Then $\vec{u} - \vec{v} \neq \vec{0} \Rightarrow T(\vec{u}) - T(\vec{v}) \neq \vec{0}$

$\Rightarrow T(\vec{u}) \neq T(\vec{v}) \Rightarrow T$ is one-to-one.

#4 Determine whether the linear transformation is one-to-one by finding its kernel and then applying Theorem 8.2.1.

a. $T : R^2 \to R^3$, where $T(x, y) = (x - y,\ y - x,\ 2x - 2y)$

b. $T : R^2 \to R^2$, where $T(x, y) = (0,\ 2x + 3y)$

c. $T : R^2 \to R^2$, where $T(x, y) = (x + y,\ x - y)$

a) $(x, y) \in \ker(T)$ if $x - y = 0,\ y - x = 0,\ 2x - 2y = 0$

$\Rightarrow x = y$. Not one-to-one.

b) $\ker(T) = \{(x, y) \mid 2x + 3y = 0\}$

$y = -\frac{2}{3}x \rightarrow$ not one-to-one

c) $x + y = 0 \Rightarrow y = 0$

$x - y = 0$

$2x = 0 \Rightarrow x = 0$

This is one-to-one.

Theorem 8.2.X If $T : V \to W$ is a one-to-one linear transformation and $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a linearly independent subset of V, then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_k)\}$ is linearly independent.

Pf: Suppose $c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + \cdots + c_k T(\vec{v}_k) = \vec{0}$.

Because T is linear, this gives

$$T(c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_k \vec{v}_k) = \vec{0}$$

Since T is one-to-one, $c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_k \vec{v}_k = \vec{0}$.

Then $c_i = 0$, $1 \le i \le k$ because $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is linearly independent.

Therefore $\{T(\vec{v}_1), T(\vec{v}_2), \dots, T(\vec{v}_k)\}$ is linearly independent.

#5 Determine whether multiplication by A is one-to-one by computing the nullity of A and then applying Theorem 8.2.1

a. $A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \\ -3 & 6 \end{bmatrix}$ b. $A = \begin{bmatrix} 1 & 3 & 1 & 7 \\ 2 & 7 & 2 & 4 \\ -1 & -3 & 0 & 0 \end{bmatrix}$

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a) $\text{rank}(A) = 1$

$\Rightarrow$ $\text{nullity}(A) = 1$

(rank + nullity = # columns)

not one-to-one.

$n = 4$, $m = 3$

$\Rightarrow$ $\text{rank}(A) \leq 3$

$\Rightarrow$ $\text{nullity}(A) \leq 1$

not one-to-one.

Theorem 8.2.2 If V and W are finite-dimensional vector spaces with the same dimension, and if $T : V \to W$ is a linear transformation, then the following statements are equivalent.

a) T is one-to-one.

b) $\ker(T) = \{\mathbf{0}\}$.

c) T is onto [i.e., $R(T) = W$].

Theorem 8.2.3 If $T_A : R^n \to R^m$ is a matrix transformation, then

a) T_A is one-to-one if and only if the columns of A are linearly independent.

b) T_A is onto if and only if the columns of A span R^m.

#14 Use Theorem 8.2.3 to determine whether multiplication by A is one-to-one, onto, both, or neither. Justify your answer.

a. $A = \begin{bmatrix} 9 & -3 \\ -4 & 2 \\ 1 & 1 \end{bmatrix}$ b. $A = \begin{bmatrix} 3 & -3 & 1 & 1 \\ 6 & -6 & 0 & 2 \\ 9 & -9 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 3 & -3 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$

$T_A : R^2 \to R^3$

$T_A : R^4 \to R^3$ $\to \begin{bmatrix} 3 & -3 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

rank is dim of range

c. $A = \begin{bmatrix} 3 & -9 \\ -1 & 3 \end{bmatrix}$

$T_A : R^2 \to R^2$

d. $A = \begin{bmatrix} 2 & 3 & 8 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

$T_A : R^3 \to R^3$

a) One-to-one. $\vec{c}_1 \neq k\vec{c}_2$.

Not onto. 2 vectors can't span R^3.

b) Not one-to-one. $\vec{c}_2 = -\vec{c}_1$. (columns are not lin indep)

Rank $(A) = 2$ but dim $R^3 = 3$. Not onto

c) Not one-to-one. $\vec{c}_2 = -3\vec{c}_1$

Not onto. (basis for Col(A) is $\left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\}$.

d) one-to-one (cols are indep)

onto (rank$(A) = 3 = $ dim $[R(T_A)]$).

Theorem 8.2.4 Equivalent Statements (extends Theorems 4.9.8, 5.1.5 and 6.4.5, the last two of which we have not seen)

If A is an $n \times n$ matrix, then the following statements are equivalent.

a) A is invertible.

b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

c) The reduced row echelon form of A is I_n.

d) A is expressible as a product of elementary matrices.

e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

g) $\det(A) \neq 0$.

h) The column vectors of A are distinct and linearly independent.

i) The row vectors of A are distinct and linearly independent.

j) The column vectors of A span R^n.
k) The row vectors of A span R^n.
l) The column vectors of A form a basis for R^n.
m) The row vectors of A form a basis for R^n.
n) A has rank n.
o) A has nullity 0.
p) The orthogonal complement of the null space of A is R^n.
q) The orthogonal complement of the row space of A is $\{\mathbf{0}\}$.
r) $\lambda = 0$ is not an eigenvalue of A. [This is from Theorem 5.1.5.]
s) $A^T A$ is invertible. [This is from Theorem 6.4.5.]
t) The kernel of T_A is $\{\mathbf{0}\}$.
u) The range of T_A is R^n.
v) T_A is one-to-one.

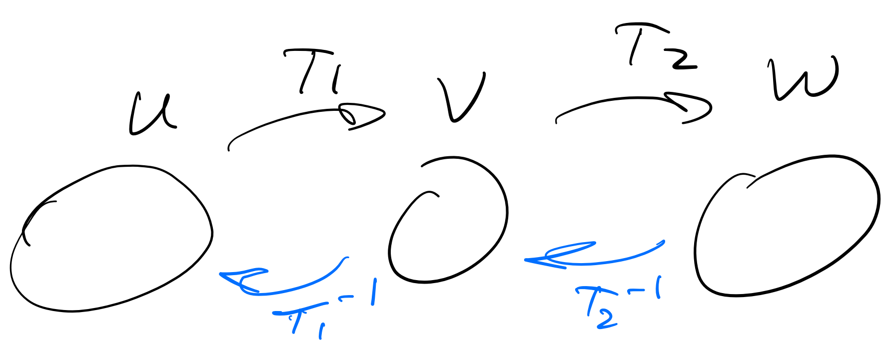
Definition: If $T : V \to W$ is a one-to-one linear transformation with range $R(T)$, and if $\mathbf{w}$ is any vector in $R(T)$, then there is exactly one vector $\mathbf{v}$ in V for which $T(\mathbf{v}) = \mathbf{w}$. The **inverse of T** maps $\mathbf{w}$ back into $\mathbf{v}$ and is denoted by T^{-1}. (This is analogous to **inverse** in Section 1.9.)

Definition: (analogous to **composition** in Section 1.9)
If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, then the **composition** of T_2 with T_1, denoted by $T_2 \circ T_1$ (which is read "T_2 circle T_1") is the function defined by the formula $(T_2 \circ T_1)(\mathbf{u}) = T_2(T_1(\mathbf{u}))$ where $\mathbf{u}$ is a vector in U.

Note that the order of the composition matters.

Theorem 8.2.5 If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, then $(T_2 \circ T_1) : U \to W$ is also a linear transformation.

Theorem 8.2.6 If $T_1 : U \to V$ and $T_2 : V \to W$ are one-to-one linear transformations, then:
a) $T_2 \circ T_1$ is one-to-one.
b) $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$.



#38 Let $D(\mathbf{f}) = f'(x)$ and $J(\mathbf{f}) = \int_0^x f(t)dt$ be the derivative and integral linear transformations. Find $(J \circ D)(\mathbf{f})$ for
a. $\mathbf{f}(x) = x^2 + 3x + 2$
b. $\mathbf{f}(x) = \sin x$

a. $J(D(x^2+3x+2)) = J(2x+3)$
$= \int_0^x (2t+3)dt = (t^2+3t\Big|_0^x$
$= x^2+3x$

b. $J(D(\sin x)) = \int_0^x \cos t\, dt = \sin t\Big|_0^x$
$= \sin x$

Key concepts from 8.3:

Definition 1: A linear transformation $T : V \to W$ that is both one-to-one and onto is said to be an **isomorphism**, and W is said to be **isomorphic** to V.

$\mathcal{F}(-\infty,\infty)$, M_{mn}, P_n

Theorem 8.3.1 Every real n-dimensional vector space is isomorphic to R^n. [In this sense, every n-dimensional vector space is algebraically equivalent to R^n.]

Theorem 8.3.2 If S is an ordered basis for a vector space V, then the coordinate map $\mathbf{u} \xrightarrow{T} (\mathbf{u})_S$ is an isomorphism between V and R^n.

Theorem 8.3.X A linear transformation $T : V \to W$ is an isomorphism if and only if whenever $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis for V, $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)\}$ is a basis for W.

Examples of Natural isomorphisms

Between P_{n-1} and R^n: $a_0 + a_1x + \ldots + a_{n-1}x^{n-1} \xrightarrow{T} (a_0, a_1, \ldots, a_{n-1})$

Between M_{22} and R^4: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{T} (a,\ b,\ c,\ d)$

Theorem 8.2.5 If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, then $(T_2 \circ T_1) : U \to W$ is also a linear transformation.

Pf:

Let T_1, T_2 be as described.

Let $\vec{x}_1, \vec{x}_2 \in U$ and $k \in \mathbb{R}$.

$$(T_2 \circ T_1)(k\vec{x}_1 + \vec{x}_2) = T_2(T_1(k\vec{x}_1 + \vec{x}_2))$$ def of composition

$$= T_2(kT_1(\vec{x}_1) + T_1(\vec{x}_2))$$ because T_1 is linear

$$= kT_2(T_1(\vec{x}_1)) + T_2(T_1(\vec{x}))$$ because T_2 is linear

$$= k(T_2 \circ T_1)(\vec{x}_1) + (T_2 \circ T_1)(\vec{x}_2)$$ def of composition